Three-points Kontsevich integral for braids

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Formal connections

Let $\nabla = d - \Omega$ be a formal connection. Formal differential 1-form

$$\Omega = \sum_{\{i,j\} \subset \{1,2,...,n\}} B_{ij} \frac{d(z_i - z_j)}{z_i - z_j}$$

is the meromorphic 1-form on $\mathbb{C}^n$ with formal coefficients B_{ij}. Forms $\omega_{ij} = \frac{d(z_i - z_j)}{z_i - z_j}$ depend only unordered pair points z_i and z_j. We suppose that $B_{ij} = B_{ji}$. As well-known constant formal coefficients B_{ij} may be interpreted as chord diagrams.

Integrability of formal connections

Frobenius condition of the integrability

$$d\Omega = \Omega \wedge \Omega$$

of formal connection ∇ is equivalent two equalities $d\Omega = 0$ and $\Omega \wedge \Omega = 0$ or to the following relations

$$[B_{ij}, B_{kl}] = 0, \text{for } \{i, j\} \cap \{k, l\} = \emptyset; \qquad (R1)$$

$$[B_{ij}, B_{jk} + B_{ik}] = 0, \text{for } i \neq j \neq k. \qquad (R2)$$

Here $[A, B] = AB - BA$. Second series of relations (R2) coincide with 4-term relations for chord diagrams B_{ij}.

Parallel transport of formal connections and Kontsevich integral

Basic ingredient of Kontsevich integral is the parallel transport

$$T_{\nabla}(\gamma) = 1 + \int_\gamma \Omega + \int_\gamma \Omega\Omega + \cdots$$

of integrable formal connection $\nabla = d - \Omega$. Here γ is a path in $\mathbb{C}^n \setminus \cup_{i \neq j}\{z_i \neq z_j\}$ presenting a braid. Summands $\int_\gamma (\Omega)^r$, $r = 1, 2, \ldots$ are the r-iterated Chen integrals

$$\int_\gamma (\Omega)^r = \sum_{p_1<q_1,\, p_2<q_2 \ldots p_r<q_r} B_{p_1,q_1} B_{p_2,q_2} \cdots B_{p_r,q_r} \int_\gamma \omega_{p_1,q_1}\omega_{p_2,q_2} \ldots \omega_{p_r,q_r},$$

where

$$\int_\gamma \omega_{p_1,q_1}\omega_{p_2,q_2}\dots\omega_{p_r,q_r} =$$

$$= \int_{\Delta^r} f_1(t_1)f_2(t_2)\cdots f_r(t_r)d\,t_1 d\,t_2 \dots d\,t_r =$$

$$= \int_0^1 (\dots(\int_0^{t_4}\int_0^{t_3}(\int_0^{t_2} f_1(t_1)d\,t_1)f_2(t_2)d\,t_2)d\,t_3\dots)f_r(t_r)d\,t_r$$

Here

$$\Delta^r = \{(t_1, t_2, \dots, t_r)|\, 0 \leq t_1 <\leq t_2 \leq \dots \leq t_r \leq 1 \subset \mathbb{R}^r\} =$$

is r-simplex in affine space $\mathbb{R}^r$ and

$$\gamma^*\omega_{p_s,q_s} = f_s(t)d\,t$$

is differential 1-form on segment $I = [0,\, 1]$.

The value of the parallel transport $T_\nabla(\gamma)$ of integrable connection ∇ depends only from the homotopic class of the path γ with fixed ends.

Parallel transport T_∇ possesses the multiplicative property

$$T_\nabla(\gamma_1\gamma_2) = T_\nabla(\gamma_1)T_\nabla(\gamma_2),$$

when the product $\gamma_1\gamma_2$ is defined.

Representations of pure braid groups in chord diagram algebras

Let $Ch_n = \mathbb{C}[B_{ij},\ 1 \leq i \neq j \leq n]/J$ be chord diagram algebra. Ideal J in the polynomial algebra of non-commutative variables $\mathbb{C}[B_{ij},\ 1 \leq i \neq j \leq n]$ is generated by terms $[B_{ij},\, B_{kl}],\ \{i,\, j\} \cap \{k,\, l\} = \emptyset$; $[B_{ij},\, B_{jk} + B_{ik}]\ i \neq j \neq k$. Denote $\mathbb{C}^n_*$ the complement $\mathbb{C}^n_* = \mathbb{C}^n \setminus \cup_{1 \leq i < j \leq n}\{z_i - z_j = 0\}$ of the $\mathbb{C}^n$ to the union of diagonal hyperplanes $z_i - z_j = 0$. Take $z_0 = (1, 2, \ldots, n) \in \mathbb{C}^n_*$. The restriction of the parallel transport T_∇ of integrable connection ∇ on the loop space $\Omega_{z_0}\mathbb{C}^n_*$ defines a representation ρ_∇ of the fundamental group $\pi_1(\mathbb{C}^n_*,\ z_0)$.

This fundamental group is isomorphic to the pure braid group P_n,

We obtain the representation

$$\rho_\nabla : P_n \longrightarrow \widehat{Ch}_n$$

in completed chord diagram algebra $\widehat{C}h_n = C << B_{ij},\ 1 \leq i \neq j \leq n >> /J.$

Since $Ker\, \rho_\nabla = 1$ (T.Kohno) then ρ_∇ is full invariant of pure braids with values in the algebra $\widehat{C}h_n$.

Three-point analog of the Kontsevich integral

V.O.Manturov proposed to consider the formal connections with forms

$$\Omega_1 = \sum_{\{i,j,k\}\subset\{1,2,...,n\}} A_{ijk}\,\omega_{ijk},$$

where meromorphic 1-forms

$$\omega_{ijk} = \frac{d(z_i - z_j)}{z_i - z_j} + \frac{d(z_j - z_k)}{z_j - z_k} + \frac{d(z_i - z_k)}{z_i - z_k}$$

depend only from unordered collection of three points z_i, z_j, z_k and A_{ijk} are formal coefficients with unordered collections of subscripts.

The integrability condition of this connection may be obtained by rewriting of the form Ω_1 as $\Omega_1 = \sum_{\{i,\,j\}\subset\{1,2,...,n\}} B_{ij}\omega_{ij}$. Here

$$B_{ij} = \sum_{r=1,\,r\neq i,\,j}^{n} A_{ijr}.$$

Then relations on A_{ijk} (*threesecant* - in Manturov terminology) are equalities

$$[\sum_{r=1,\,r\neq i,\,j}^{n} A_{ijr}, \sum_{m=1,\,m\neq k,\,l}^{n} A_{klm}] = 0,\ \{i,\,j\}\cap\{k,\,l\} = \emptyset. \quad (R3)$$

$$[\sum_{p=1,\,p\neq i,\,j}^{n} A_{ijp}, \sum_{q=1,\,q\neq j,\,k}^{n} A_{klq} + \sum_{r=1,\,r\neq i,\,k}^{n} A_{klr}] = 0,\ i \neq j \neq k. \quad (R4)$$

In particular case $n = 4$ we have relations:

$$[A_{123} + A_{124}\,,\ A_{134} + A_{234}] = 0;$$

$$[A_{123} + A_{134}\,,\ A_{124} + A_{234}] = 0;$$

$$[A_{124} + A_{134}\,,\ A_{123} + A_{234}] = 0;$$

$$[A_{123} + A_{124}\,,\ 2A_{123} + A_{134} + A_{234}] = 0;$$

$$[A_{123} + A_{124}\,,\ 2A_{124} + A_{134} + A_{234}] = 0,$$

$$[A_{123} + A_{234}\,,\ 2A_{234} + A_{124} + A_{134}] = 0;$$

$$[A_{123} + A_{234}\,,\ 2A_{123} + A_{134} + A_{234}] = 0.$$

Representation of the P_n in threesecant algebra

Define the threesecatnt algebra as quotient of algebra non-commutative polynomials

$$TS_n = \mathbb{C}[A_{ijk},\ \{i,\, j\ k\} \subset \{1,\, 2,\, \ldots,\, n\}]/J_1,$$

where J_1 is ideal generated right sides of relations (R_1) and (R_2). The completed the threesecatnt algebra is following quotient-algebra

$$\widehat{TS}_n = \mathbb{C} << A_{ijk},\ \{i,\, j\ k\} \subset \{1,\, 2,\, \ldots,\, n\} >> /J_1,$$

Parallel transport T_{∇_1} of the integrable connection $\nabla_1 = d - \Omega_1$ with form Ω_1 under restriction on the loop space $\Omega_{z_0}\mathbb{C}^n_*$ defines a representation ρ_{∇_1} of pure braid group P_n in the completed threesecant algebra, i.e the associative algebra of series from non-commutative variables A_{ijk} with relations (R_1) and (R_2)

$$\rho_{\nabla_1} : P_n \longrightarrow \widehat{TS}_n.$$

Let a path $\gamma \in \Omega_{z_0}\mathbb{C}^n_*$ defines a braid $b \in P_n$ We call the value $T_{\nabla_1}(\gamma) = \rho_{\nabla_1}(b)$ the three-point Kontsevich integral for braid b.

Generators and relations of pure braid group P_n

Let L be some line in affine space $\mathbb{C}^n$, intersecting all hyperplanes $H_{ij} = \{z \in \mathbb{C}^n | z_i = z_j\}$ in general position. Denote $p_{ij} = l \cap H_{ij} \in L$ points of intersection L and H_{ij}, $1 \leq i < j \leq n$. Take loops γ_{ij} on L with initial point $p \neq p_{ij}, 1 \leq i < j \leq n$, l single around about p_{ij}. We suppose that loops $\gamma_{ij} \cap \gamma_{rs} = p$, $\{i, j\} \neq \{r, s\}$. Loops γ_{ij}, $1 \leq i < j \leq n$ present a system of free generators b_{ij} in free group $\pi_1(L \setminus \{p_{ij},\ 1 \leq i < j \leq n\}, z^0)$ and a system of generators b_ij in pure braid groups $PB_n = \pi_1(\mathbb{C}^n_*, z_0 = (1, 2, \ldots, n))$. Relations in PB_n correspond loops in a transversal plane to the line L. These loops go around about points of intersection of hyperplanes H_{ijk} and H_{ijkl}.

Generators and relations of Birman-Ko-Lee:

$$b_{ij}b_{kl} = b_{kl}b_{ij},\ i<j<k<l \text{ или } i<k<l<j,$$

$$b_{ij}b_{ik}b_{jk} = b_{ik}b_{jk}b_{ij},\ i<j<k$$

$$b_{jl}b_{kl}b_{ik}b_{jk} = b_{jk}b_{ik}b_{kl}b_{jl},\ i<j<k<l.$$

Artin generators and Birman-Ko-Lee generators

$$b_{ij} = A_{i,i+1}A_{i+1,i+2}\cdots A_{j-1,j}.$$

Manturov group G_n^3

The Manturov group G_n^3 is defined as a group with generators a_{ijk}, $\{i,\, j,\, k\} \subset \{1,\, 2,\, \ldots,\, n\}$ and relations

$$a_{ijk}^2 = 1,$$

$$a_{ijk}a_{pqr} = a_{pqr}a_{ijk}, \text{ if } |\{\text{ß},\, j,\, k\} \cap \{p,\, q,\, r\}| \leq 1,$$

$$(a_{jkl}a_{ikl}a_{ijl}a_{ijk})^2 = 1, \text{ for } \forall\{i,j,k,l\} \subset \{1,\, 2,\ldots,\, n\}.$$

Manturov-Nikonov representation of pure braid groups

Manturov-Nikonov defined the homomorphism φ_n of the pure braid P_n in the group G_n^3 by values on generators b_{ij}

$$\varphi_n(b_{ij}) = c_{ii+1}^{-1} \dots c_{ij-1}^{-1} c_{ij}^2 c_{ij-1} \dots c_{ii+1}, \qquad (1)$$

где

$$c_{ij} = \prod_{k=j+1}^{n} a_{ijk} \prod_{k=1}^{j-1} a_{ijk}. \qquad (2)$$

Realization of $\varphi_n(b_{ij})$ as values parallel transport

Let $\gamma_{ij} \in \Omega_{z_0}\mathbb{C}^n_*$ be loops presenting generators b_{ij} of the P_n. There are exist a thresecants A_{ijk} in the group algebra $\mathbb{C}[G_n^3]$ is the group algebra of the Manturov group G_n^3, such that we have equalities $T_{\nabla_1}(\gamma_{ij}) = varphi_n(b_ij)$.

Lappo-Danilevskii inversion method

Let us consider the group algebra $\mathfrak{g}_n^3 = \mathbb{C}[G_n^3]$ as Lie algebra with respect ordinary Lie bracket $[a,b] = ab - ba$. Universal enveloping algebra $U(\mathfrak{g}_n^3)$ is isomorphic to $\mathbb{C}[G_n^3]$. Let $\widehat{\mathbb{C}[G_n^3]}$ be the completion of the $\mathbb{C}[G_n^3]$ by the augmentation ideal и $J \subset \mathbb{C}[G_n^3]$, generated elements a $g-1$, $g \in G_n^3$. We will look for B_{ij} in the ideal $\hat{J} \subset \widehat{\mathbb{C}[G_n^3]}$.

Consider the system of equations

$$\varphi_n(b_{ij}) = T_{\nabla}(\gamma_{ij})$$

or

$$c_{i,i+1}^{-1} \dots c_{i,j-1}^{-1} c_{ij}^{2} c_{i,j-1} \dots c_{i,i+1} = 1 + \int_{\gamma_{ij}} \Omega + \int_{\gamma_{ij}} \Omega\Omega + \cdots + \int_{\gamma_{\gamma_{ij}}} \Omega^{m} + \cdots .$$

Rewritten last system

$$c_{i,i+1}^{-1} \dots c_{i,j-1}^{-1} (c_{ij}^{2} - 1) c_{i,j-1} \dots c_{i,i+1} = \sum_{p<q} B_{pq} \int_{\gamma_{ij}} \omega_{pq} + \sum_{p<q,\, r<s} B_{pq} B_{rs} \int_{\gamma_{ij}} \omega_{pq}$$

$$\cdots + \sum_{p_1<q_1,\, p_2<q_2 \dots p_m<q_m} B_{p_1,q_1} B_{p_2,q_2} \cdots B_{p_m,q_m} \int_{\gamma_{ij}} \omega_{p_1,q_1} \omega_{p_2,q_2} \dots \omega_{p_m,q_m} + \cdots .$$

Here $\omega_{pq} = \frac{d(z_p - z_q)}{z_p - z_q}$.

By theorem about inverting formal series (then the matrix of the linear part is invertible) we obtain series for B_{pq}, $p < q$ as series from left parties $M_{ij} = c_{i,i+1}^{-1} \dots c_{i,j-1}^{-1}(c_{ij}^2 - 1)c_{i,j-1} \dots c_{i,i+1}$ of equations above.

Integrability of the formal connections

Integrability of the connection $\nabla = d - \Omega$ may extracted from generating commutator relations of the P_n.

We obtain the formal connection $\nabla_1 = d - \Omega_1$ solving the system of the linear equations

$$B_{ij} = \sum_{r=1,\, r \neq i,\, j}^{n} A_{ijr}.$$

The integrability of the ∇ implies the integrability of the ∇_1.

Thus we obtain realization

$$\varphi_{(}b_{ij}) = \rho_{\nabla_1}(b_{ij})$$

of the representation φ_n as the monodromy representation of a integrable connection.

References

[1]. N.Bourbaki, *Groupes et algèbres de Lie.* Paris: Herman, 1968.

[2]. C. Kassel, V. Turaev, *Braid groups.* Springer-Verlag. 2008.

[3]. T. Kohno, *Linear representations of braid groups and classical Yang-Baxter equations.* Contemporary Math. 1988. V.78. P. 339–363.

[4]. I. A. Lappo-Danilevskii, *Application of functions from matrices to the theory of systems of ordinary linear differential equations Russian translation.* Mir, GITTL 1957. 456 p.

[5]. V. P. Leksin, *Riemann-Hilbert problem for analytic families of representations.* Mathematical Notes. 1991. T. 50. №. 2. P. 832-838.

[6]. W. Magnus, A. Karrass, D. Solitar, *Combinatorial group theory. Presentation of groups by generators and relations.* New York – London – Sydney: Interscience Publisher, John Wiley and Sons, Inc. 1966.

[7]. O. V. Manturov, *Multiplicative integral.* Itogi nauki i tekhn.. Ser. Probl. geom.. T. 22, VINITI, M., 1990, 167–215.

[8]. V. O. Manturov, *Non-reidemeister knot theory and its applications in dynamical systems, geometry and topology.* arXiv:

1501.05208 [math.GT].

[9]. V. O. Manturov and I. M. Nikonov, *On braids and groups G_n^3.* arxiv: 1507.03745[math.GT].

[10]. R. Hain *Hain R. M. Iterated integrals and homotopy periods.* – American Mathematical Soc., 1984. – T. 291.